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Minimal model boundary flows and $c\!=\!1$ CFT

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Abstract

We consider perturbations of unitary minimal models by boundary fields. Initially we consider the models in the limit as $c \to 1$ and find that the relevant boundary fields all have simple interpretations in this limit. This interpretation allows us to conjecture the IR limits of flows in the unitary minimal models generated by the fields ϕ_{rr} of 'low' weight. We check this conjecture using the truncated conformal space approach. In the process we find evidence for a new series of integrable boundary flows.

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1   Introduction

We consider perturbations of unitary minimal models by boundary fields. Perturbations of these models by the least relevant boundary field have been studied by Recknagel et al. [23], in perturbation theory around $c=1$. One simple observation is that all the perturbative flows they find must collapse to identities at $c=1$, (i.e. the start and end points of the flows for $c<1$ must be identified at $c=1$) and so one might hope to gain some insight into flows for $c<1$ generally by examining the structure of the $c \to 1$ limit of the boundary minimal models. In particular, non-perturbative flows are hard to understand in absence of a Landau Ginzburg picture, and they may be easier to understand as 'perturbations' of genuine $c=1$ flows. This paper provides arguments to support this point of view, which was outlined in [17].

In section 2 we consider the unperturbed boundary conditions (b.c.'s). After some generalities, we review the 'A' type unitary minimal models and their b.c.'s which we shall denote $(r,s) \equiv B_{(r,s)}$. Of these, $(1,r)$ and $(r,1)$ play a special role in the $c \to 1$ limit, and the two b.c.'s become identified as a new boundary condition which we denote by $(\hat{r})$. We derive various properties of the $(\hat{r})$ b.c.'s. We can then formulate our main conjecture, that in the limit $c \to 1$, a general boundary condition of type (r,s) splits up into a superposition of min (r,s) such 'fundamental' b.c.'s:

$$\lim_{c \to 1} (r,s) = \oplus_{n=1}^{\min(r,s)} \left(\widehat{|r-s|+2n-1} \right) . \tag{1.1}$$

Superpositions of boundary conditions have already been seen in the form of Chan-Paton factors in string theory (see e.g. [22]) and also in the work of Affleck [1] and in [23]. The superpositions generically contain multiple boundary fields of weight zero. In our case we can investigate the properties of these weight zero fields explicitly since they are the $c \to 1$ limit of the primary fields ϕ_{rr} in the unitary minimal models, and we know all their structure constants from [24].

The weight zero and weight one fields on a boundary with b.c. (1.1) are of special interest since these include the limits of all the relevant boundary fields for $c<1$. We argue that the weight zero fields on the $c \to 1$ limit of the (r,s) b.c. can be decomposed into linear combinations of projectors onto the fundamental boundary conditions $(\hat{t})$ appearing in (1.1). The weight one fields can then be divided into fields which act solely within the fundamental sectors, and fields which interpolate pairs of them (i.e. boundary condition changing operators). We present an explicit analysis of the $(2,2)$ boundary condition in section 3.3, and for the series of boundary conditions $(2,p)$ and $(3,p)$ in appendix A.

In section 4 we turn to boundary perturbations, and start with the case of $c=1$. Since the weight zero fields can be expressed in terms of projectors, perturbation by these fields becomes easy to understand. We then turn to the models with $c<1$. In section 4.1 we examine the perturbations by the fields of type ϕ_{rr} using the truncated conformal space approach (TCSA) and find that we are indeed able to predict the IR end-points from our analysis of the corresponding flows at $c=1$ (up to the ambiguity $(r,s) \leftrightarrow (s,r)$ in the boundary conditions). In the process, we find strong evidence for the integrability of the perturbation of the model $M_{r+1,r+2}$ by the particular boundary field $\phi_{rr} \equiv \phi_{12}$.

The perturbations of the $c=1$ model by the weight one fields is more difficult and we defer this to a later paper [18], along with a discussion of perturbations of the minimal models by

boundary fields of type $\phi_{r,r+2}$. We conclude with some discussion of these results and possible extensions.

2   The boundary conditions of the unitary minimal models

The original papers by Cardy and Lewellen [8, 9, 20] on boundary conformal field theory set out the basic properties – the boundary field content, and the consistency conditions satisfied by the various structure constants. The boundary field contents of all Virasoro minimal models were found in [4–6], and a full solution of these consistency conditions for the A–type Virasoro minimal models was proposed in [24]. For more recent developments, see [14, 21]. First we give some general results on boundary models and then discuss the minimal models for $c < 1$.

2.1   Some generalities

Consider the upper half plane (UHP) with the boundary condition α on the left real axis and β on the right real axis. A single copy of the Virasoro algebra acts on the upper half plane, and so the Hilbert space $\mathcal{H}_{\alpha\beta}$ of the upper half plane with this pair of boundary conditions splits into a direct sum of irreducible representations R_c of the Virasoro algebra[*],

$$\mathcal{H}_{\alpha\beta} = \sum_c n_{\alpha\beta}{}^c R_c , \tag{2.1}$$

where the numbers $n_{\alpha\beta}{}^c$ should be non-negative integers; if we further impose the condition that the identity representation $R_{\mathbb{1}}$ appear at most once, we call the resulting subset of boundary conditions 'fundamental'.

The states in the Hilbert $\mathcal{H}_{\alpha\beta}$ are in one-to-one correspondence with the fields which interpolate the boundary conditions α and β. In particular, the fields which can lie on the α boundary are in one-to-one correspondence with the states in $\mathcal{H}_{\alpha\alpha}$, and the space of primary fields of type c which live on the boundary α has dimension $n_{\alpha\alpha}{}^c$.

The UHP can be related to an infinite strip of width R by a conformal transformation, and the Hamiltonian generating translations along the strip is (in terms of the Virasoro algebra on the UHP)

$$H(R) = (\pi/R)(L_0 - c/24) . \tag{2.2}$$

Hence the partition function on a cylinder of width R and circumference L with b.c.'s (α, β) on the two edges is

$$Z_{\alpha\beta}(L, R) \equiv \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left(e^{-LH(R)} \right) = \sum_c n_{\alpha\beta}{}^c \chi_c(q) , \tag{2.3}$$

where $\chi_c(q)$ are the characters of the irreducible Virasoro highest weight representations R_c

$$\chi_c(q) = \mathrm{Tr}_{R_c}\left(q^{L_0 - c/24} \right) , \quad q = \exp(-\pi L/R) . \tag{2.4}$$

Hence from equations (2.1) and (2.3) we see that the cylinder partition functions encode the boundary field content.

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[*]In general this can only be proven for unitary models

2.2 Minimal models

From now on we shall assume that we are dealing with the A–type Virasoro minimal models (for general properties of the minimal models, see e.g. [10]). Each model $M(p, p')$ is labelled by two positive coprime integers $p, p' > 1$, or alternatively by the rational number $t = p/p'$. Associated to each model is a set of $(p-1)(p'-1)/2$ Virasoro representations R_i labelled by i, given by a set of weights $\{h_i\}$. The central charge c and the weights $h_{rr'}$ take the form

$$c = 13 - 6t - 6/t , \quad h_{rr'} = \frac{1}{4t}((r - r't)^2 - (1-t)^2) , \tag{2.5}$$

where the Kac-labels (r, r') lie in the ranges $r = 1..p-1$ and $r' = 1..p'-1$. If we allow all pairs (r, r') in these ranges then each weight appears twice, since $h_{rr'} = h_{p-r, p'-r'}$.

We can take a definite choice of representatives of the Kac labels (r, r') as follows. At least one of p and p' is odd. Suppose p is odd. Then the pairs $\{ (r, r'), 1 \le r \le p - 2, 1 \le r' \le p' - 1, r \text{ odd} \}$ run over the set of Virasoro representations once and once only.

The fusion product of the representations R_a is described by the Verlinde algebra

$$R_a \times R_b = \sum_c N_{ab}{}^c R_c , \tag{2.6}$$

where (in this case) the Verlinde fusion numbers $N_{ab}{}^c$ are either 0 or 1. With the choice of representatives given above, the fusion numbers are explicitly

$$N_{(rr')(ss')}{}^{(tt')} = \mathcal{N}_{rs}^t(p) \cdot \mathcal{N}_{r's'}^{t'}(p')$$
$$\mathcal{N}_{ab}^c(m) = \begin{cases} 1 : & |a-b| < c < \min(a+b, 2m-a-b) , \ a+b+c \text{ odd} \\ 0 : & \text{otherwise} \end{cases} \tag{2.7}$$

The characters of the minimal model representations are

$$\chi_{(r,r')}(q) = \frac{q^{-c/24}}{\varphi(q)} \sum_{n=-\infty}^{\infty} \left(q^{h_{(r+2np),r'}} - q^{h_{(r+2np),-r'}} \right) , \quad \varphi(q) = \prod_{n=1}^{\infty}(1 - q^n) . \tag{2.8}$$

Their behaviour under modular transformations $q = e^{2\pi i\tau} \to \tilde{q} = e^{-2\pi i/\tau}$ is given by the matrix

$$S_{rr'}{}^{ss'} = 2^{3/2}(pp')^{-1/2}(-1)^{1+rs'+r's} \sin(\pi rs/t) \sin(\pi r's't) . \tag{2.9}$$

2.3 Boundary conditions

For the A–type minimal models, the fundamental boundary conditions are in 1–1 correspondence with the representations of the Virasoro algebra and so we can label both boundary conditions and representations from the same set $\{a\}$; in this case the numbers $n_{ab}{}^c$ are the Verlinde fusion numbers [8]. Since the fundamental boundary conditions a are in one-to-one correspondence with the set of Virasoro representations, we will denote them by a, h_a or (r_a, r'_a) interchangeably.

It is convenient to define an ordering on the boundary conditions, for example

$$(r, r') > (s, s') \iff (r' > s' \text{ or } (r' = s' \text{ and } r > s)) . \tag{2.10}$$

We choose to normalise the primary fields and one-point functions of the boundary theory so that for boundary conditions a, b and a primary boundary field i we have

$$\langle 1 \rangle^a = S_1{}^a / S_1{}^1 , \quad a \le b : \ C_{ii}^{(aba)1} = 1 \tag{2.11}$$

In particular this implies that for $a > b$, $C_{ii}^{(aba)1} = S_1{}^b / S_1{}^a$ [20, 24].

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3 The $c \to 1$ limit of the boundary unitary minimal models

To study the limit $c \to 1$, we set $t = 1 - \epsilon$, so that the central charge $c \sim 1 - 6\epsilon^2$. We shall denote characters, partition functions, Hilbert spaces, etc, at $c=1$ by $\widehat{\chi}$, $\widehat{Z}$, $\widehat{\mathcal{H}}$, etc, to distinguish them from those in the minimal models with $c<1$. We first recall the result of Recknagel et al. [23] which led to our conjecture.

In [23], they studied the renormalisation group flows of a boundary condition $(\alpha) = (a, a')$ generated by the field $\phi_{13}^{(\alpha\alpha)}$ by the addition to the action of the integral along the boundary,

$$\delta S = \lambda \oint dl \ \phi_{13}^{(\alpha\alpha)} . \tag{3.1}$$

They calculated the perturbative beta function for λ for $\epsilon \sim 0$ and found an attractive fixed point for a value $\lambda^* = O(\epsilon)$. They examined the properties of this fixed point and found that it could only be described by a superposition of b.c.'s,

$$(a, a') \xrightarrow[\lambda \to \lambda^*]{} \oplus_{n=1}^{\min(a,a')} (\,|a-a'|+2n-1, 1\,) . \tag{3.2}$$

Since the fixed point is at $\lambda^* = O(\epsilon)$, as $c \to 1$ (and consequently $\epsilon \to 0$) the fixed point moves closer to $\lambda = 0$, and in the limit this flow must degenerate to an identity. Denoting the limit as $c \to 1$ of the b.c. $(r, 1)$ by $(\widehat{r})$, we are led to our main conjecture:

Conjecture 1

The $c \to 1$ limit of the boundary condition $\alpha = (a, a')$ is the superposition of $\min(a, a')$ 'fundamental' boundary conditions, which we represent schematically as

$$\lim_{c \to 1} (a, a') = \oplus_{n=1}^{\min(a,a')} (\,|a-\widehat{a'|+2n}-1\,) . \tag{3.3}$$

That is, the field content and the correlation functions for the (a, a') boundary condition are identical (in the $c \to 1$ limit) to those on the superposition of the fundamental boundary conditions.

An important consequence of this conjecture is that the boundary fields that arise on the (r, s) b.c. in the $c \to 1$ limit can be expressed in terms of the boundary fields of the superposition. In particular, the scalar (weight zero) fields that arise must be spanned by the projectors onto the various fundamental components of the superposition (3.3), and the weight one fields must split into boundary fields living on a single fundamental component and b.c.-changing fields which interpolate two different fundamental b.c.'s. This leads to:

Conjecture 2

The operator product algebra B of the scalar fields on the boundary condition $\alpha = \oplus\widehat{\alpha}_i$ is the algebra of projectors P_i onto the fundamental components $\widehat{\alpha}_i$ of the boundary condition α. In particular, if we consider the upper half plane, the scalar fields on the left and right of the origin generate two commuting copies of this algebra, B^L and B^R, which act on the Hilbert space $\widehat{\mathcal{H}}(\ \underline{\alpha \curvearrowright \alpha}\ )$, and the projectors in these two algebras project onto the subspaces

$$P_i^L P_j^R : \widehat{\mathcal{H}}(\ \underline{\alpha \curvearrowright \alpha}\ ) \longrightarrow \widehat{\mathcal{H}}(\ \underline{\widehat{\alpha}_i \curvearrowright \widehat{\alpha}_j}\ ) . \tag{3.4}$$

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We have not been able to prove these conjectures, but have checked them quite extensively. In the next section we show that the $c \to 1$ limits of the cylinder partition functions are in agreement with conjecture 1, and in section 3.3 we find explicitly the relation between the weight zero and weight one boundary fields on the b.c. $(2,2)$ and the superposition $\widehat{(1)} \oplus \widehat{(3)}$, and show that they are in accord with conjecture 2. We have also examined the general $(2,p)$ and $(3,p)$ boundaries, and leave these results to the appendix in sections A.3 and A.4.

3.1 The $c \to 1$ limit of the cylinder partition functions

The limit $\epsilon \to 0$ corresponds to picking a sequence of minimal models such that the ratio p/p' approaches 1. For the bulk models this poses some problems, since the bulk theory has $(p-1)(p'-1)/2$ fields, which clearly tends to infinity as $c \to 1$. However, the boundary theories are rather better behaved in this respect.

One important feature of the limit $c \to 1$ which we note here is that for p and p' large enough, the truncated fusion rules (2.7) are replaced by simple $su(2)$ fusion rules:

$$\widehat{\mathcal{N}}_{ab}{}^c \equiv \lim_{m\to\infty} \mathcal{N}_{ab}{}^c(m) = \begin{cases} 1: & |a-b| < c < (a+b) \ , \ a+b+c \ \text{odd} \\ 0: & \text{otherwise} \end{cases} \tag{3.5}$$

Let us now consider one or more particular fixed boundary conditions $\alpha=(a,a')$, $\beta=(b,b')$, etc. For fixed boundary conditions α, β and p and p' large enough, the untruncated fusion rules (3.5) mean we can write the partition function $Z_{(\alpha,\beta)}$ as

$$Z_{(a,a'),(b,b')} = \sum_{\substack{c \in a \otimes b \\ c' \in a' \otimes b'}} \chi_{(c,c')} \ , \tag{3.6}$$

where $c \in a \otimes b$ is a shorthand notation to indicate that the sum runs over all labels c that occur in the tensor product of the $su(2)$ representations a and b. Hence, for p and p' large enough, there are $\min(a,b) \cdot \min(a',b')$ primary fields interpolating the boundary conditions α and β, and in particular the boundary condition α has a fixed boundary field content of $a \cdot a'$ fields. Furthermore, for p,p' large enough, the number of states (up to any particular level) becomes constant, and we can hope that any particular physical quantity in the theory (structure constant, correlator, etc) will approach a limiting value as $c \to 1$. We cannot prove that this limit is well defined, but examination of several cases suggests that this is likely to be the case. As a first step to finding this limit, we can find the field content from the strip partition functions.

While the partition function itself has a smooth limit as $c \to 1$, the decomposition of the Hilbert space into irreducible representations of the Virasoro algebra does not. The essential point is that the $\epsilon \to 0$ limit of the minimal model character $\chi_{rr'}$ with fixed r, r' is not, in general, the character of an irreducible $c=1$ representation. We shall discuss in section 3.1.1 how it is that the fields and states in a single irreducible representation for $c<1$ can reassemble themselves into several irreducible representations at $c=1$. For the moment we shall assume this works, and present the results.

The relevant irreducible highest-weight representations at $c=1$ are labelled by a single positive integer (r) with weights and characters

$$\widehat{h}_r = \frac{(r-1)^2}{4} \ , \quad \widehat{\chi}_r = \frac{q^{h_r-1/24}}{\varphi(q)} \left(1 - q^r \right) . \tag{3.7}$$

In terms of these, we have

$$\lim_{c \to 1} h_{rr'} = \widehat{h}_{|r-r'|+1} \,, \qquad \lim_{c \to 1} \chi_{(r,r')} = \sum_{n=1}^{\min(r,r')} \widehat{\chi}_{|r-r'|+2n-1} \,, \qquad (3.8)$$

and in the limit $c \to 1$ the representations $R_{(r,r')}$ and $R_{(r',r)}$ are identical. In particular,

$$\lim_{c \to 1} h_{1,r} = \lim_{c \to 1} h_{r,1} = \widehat{h}_r \,, \quad \lim_{c \to 1} \chi_{1,r} = \lim_{c \to 1} \chi_{r,1} = \widehat{\chi}_r \,, \qquad (3.9)$$

and so the representations $R_{(r,1)}$ and $R_{(1,r)}$ both have as their limit the single irreducible representation $R_{(\widehat{r})}$, justifying our notation $\lim_{\epsilon \to 0}(r,1) = (\widehat{r})$.

Note that the decomposition (3.8) is given by the same $su(2)$ fusion rules that appear in (3.5), so that we can just as well write

$$\lim_{c \to 1} \chi_{(r,r')} = \sum_{s \in r \otimes r'} \widehat{\chi}_s \,. \qquad (3.10)$$

Applying this to (3.6), we find

$$\lim_{c \to 1} Z_{(a,a'),(b,b')} = \sum_{\substack{c \in a \otimes b \\ c' \in a' \otimes b'}} \lim_{c \to 1} \chi_{(c,c')} = \sum_{d \in a \otimes b \otimes a' \otimes b'} \widehat{\chi}_d = \sum_{\substack{e \in a \otimes a' \\ e' \in b \otimes b'}} \sum_{d \in e \otimes e'} \widehat{\chi}_d = \sum_{\substack{e \in a \otimes a' \\ e' \in b \otimes b'}} \widehat{Z}_{(\widehat{e})(\widehat{e}')} \,, \quad (3.11)$$

where in the last equality we used equation (2.3) as applied to the $c{=}1$ fusion rules (3.5). In other words, if we write the characters appearing in the decomposition as

$$\lim_{c \to 1} \chi_a = \sum_{\{a_i\}} \widehat{\chi}_{a_i} \,, \qquad (3.12)$$

then the partition function for the cylinder with boundary conditions (α, β) satisfies

$$\lim_{c \to 1} Z_{\alpha,\beta} = \sum_{\{\alpha_i\},\{\beta_j\}} \widehat{Z}_{\alpha_i,\beta_j} \,, \qquad (3.13)$$

which is in exact agreement with the conjecture (3.3).

3.1.1 The 'missing' fields

As explicitly shown in (3.10), the limit of an irreducible representation of the Virasoro algebra for $c{<}1$ need not be an irreducible representation at $c{=}1$. This is due to to the fact that certain vectors may become null as $c \to 1$. Consider the representation $h_{rr'}$. For $c{<}1$ and p, p' large enough, the first null vector in the representation occurs at level $r{\cdot}r'$. However, at $c{=}1$, $h_{rr'} = \widehat{h}_{|r-r'|+1}$, and the first null vector in this representation occurs at level $l = (|r-r'|+1)$. Since we require the number of fields of a given weight not to change abruptly as $c \to 1$, we need to find a way to keep this state in the spectrum. The solution is simply to normalise the state to unit norm so that it does not decouple, and normalise the corresponding field accordingly. This may lead to divergent correlation functions involving this new field, but

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we find in practice that this is not the case. For example, consider the case of the (33) representation. We have

$$\lim_{c \to 1} h_{33} = 0 , \quad \lim_{c \to 1} \chi_{33} = \widehat{\chi}_1 + \widehat{\chi}_3 + \widehat{\chi}_5 . \tag{3.14}$$

We see that at $c=1$ there arise two new primary fields of weight $\widehat{h}_3 = 1$ and $\widehat{h}_5 = 4$, and correspondingly there are also null states at level 1 in the representation with $h = 0$, at level 3 in the representation with $h = 1$, and at level 5 in the representation with $h = 4$. Concentrating on the representation with $h = 0$, the new null state at level 1 is simply

$$L_{-1} | h_{33} \rangle . \tag{3.15}$$

If we now consider the same state for $c<1$, we find that its norm is

$$\langle h_{33} | L_1 L_{-1} | h_{33} \rangle = 2h_{33} = 4\epsilon^2/t , \tag{3.16}$$

and it will decouple from all correlation functions. However, if we define the state

$$| d_3 \rangle = \lim_{\epsilon \to 0} \frac{\sqrt{1-\epsilon}}{2\,\epsilon} L_{-1} | h_{33} \rangle , \tag{3.17}$$

this new state has unit norm, $\langle d_3 | d_3 \rangle = 1$, and has all the properties we require; in particular it is a highest weight state, since it is annihilated by L_m with $m > 0$. For example, the action of L_1 is

$$L_1 \left[\frac{\sqrt{1-\epsilon}}{2\epsilon} L_{-1} | h_{33} \rangle \right] = \frac{\sqrt{1-\epsilon}}{2\epsilon} (2h_{33}) | h_{33} \rangle = 2\epsilon | h_{33} \rangle + O(\epsilon^2) , \tag{3.18}$$

so that $L_1 | d_3 \rangle = 0$. We expect all the extra required primary fields to arise in this way, and their correlation functions to be well defined in the $c \to 1$ limit. For example, the opes of d_3 with the fields of weight 0 and 1 on the $(2, 2)$ b.c. are calculated in the appendix and shown to be regular.

3.2   The scalar fields of weight 0

For this section we will restrict ourselves to the case of the upper half plane with a single boundary condition $\alpha = (a, a')$. For general c, the Hilbert space $\mathcal{H}_{\alpha\alpha}$ splits as

$$\mathcal{H}_{\alpha\alpha} = \sum_k N_{\alpha\alpha}{}^{\kappa} R_{\kappa} ,$$

so that only representations κ with $N_{\alpha\alpha}{}^{\kappa}$ non-zero occur. In particular, this implies that the Kac labels of the representation $\kappa = (k, k')$ will both be odd. Since $\lim_{c \to 1} h_{rr'} = (r - r')^2/4$, in the limit $c \to 1$ all the weights h_κ of the primary boundary fields will be integers, and hence all the states in $\mathcal{H}_{\alpha\alpha}$ will have integer weight. If we let $\mathcal{H}^{(k)}, k = 0, 1, \ldots$ be the subspace of $\mathcal{H}_{\alpha\alpha}$ of L_0 eigenvalue k, i.e. the space of all fields of weight k, then

$$\mathcal{H}_{\alpha\alpha} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \ldots \tag{3.19}$$

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Each state in (3.19) corresponds to a field on the boundary, and those in $\mathcal{H}^{(0)}$ of weight 0 are of special interest. If $|\psi_i\rangle$ are states of weight 0, and $\psi_i(z)$ the corresponding field, then the action of the Virasoro algebra on such fields is[†]

$$[L_m, \psi_i(z)] = z^{m+1} \frac{\mathrm{d}\psi_i}{\mathrm{d}z} \ , \tag{3.20}$$

In a unitary theory, the field $\mathrm{d}\psi_i/\mathrm{d}z$ corresponds to the state $L_{-1}|0\rangle$ which is a null state, and so can be set to zero in all correlation functions. Hence, in a unitary theory, we have

$$[L_m, \psi_i(z)] = 0 \ , \tag{3.21}$$

for all fields of weight 0. Since we can obtain the states in $\mathcal{H}^{(0)}$ through a sequence of unitary models, we expect that the fields will obey (3.21) in the limit $c \to 1$. (Note that (3.21) need not always be true for fields of weight 0 – a counter example is percolation where fields of weight 0 can have non-trivial space dependence for both $c = 0$ [7, 26] and $c = 1/2$ [19].)

The fact that the fields ψ_i are scalars under local conformal transformations means that they preserve the grading in (3.19). In other words, the ope of such a scalar field with a primary field of weight h is again a primary field of weight h. However, while the fields of weight 0 are scalars under local conformal transformations, such transformations cannot alter the order of fields along a boundary. This means that the fields of weight 0 do not have to commute with the boundary fields. To be explicit, if ψ_i are the fields in $\mathcal{H}^{(0)}$, and those in $\mathcal{H}^{(k)}$ are denoted by Ψ_j, their opes take the form

$$
\begin{aligned}
\psi_i(x)\,\Psi_j(0) &= \sum_k C_{ij}^{(+)k}\,\Psi_k(0) \ , & x &> 0 \ , \\
\Psi_j(0)\,\psi_i(x) &= \sum_k C_{ij}^{(-)k}\,\Psi_k(0) \ , & x &< 0 \ ,
\end{aligned}
\tag{3.22}
$$

where $C_{ij}^{(+)k}$ and $C_{ij}^{(-)k}$ need not be equal. For the A–type models, the couplings between *primary* boundary fields are symmetric as shown in [24], but for $k > 0$, by no means all the fields in $\mathcal{H}^{(k)}$ arise as the limits of primary fields.

In the particular case of $\mathcal{H}^{(0)}$, however, all these fields arise as the limits of primary fields for $c < 1$, so that in this case $C_{ij}^{(+)k} = C_{ij}^{(-)k}$ and the fields of weight 0 do commute amongst themselves. Hence the operator product algebra of the weight 0 fields simplifies to a straightforward finite-dimensional, commutative, associative algebra:

$$\psi_i \cdot \psi_j = b_{ij}^k \, \psi_k \ .$$

The space $\mathcal{H}_0$ consists precisely of (the limits of) all primary boundary fields with Kac labels (r, r) such that $N^{(r,r)}_{(a,a')(a,a')} \neq 0$, i.e.

$$\mathcal{H}^{(0)} = \{ \ \phi_{rr} \mid r = 1, 3, 5, \ldots, 2\min(a, a') - 1 \ \} \ . \tag{3.23}$$

Since we take all fields on a given boundary α to have unit norm (2.11), the numbers b_{ij}^k are the limits of the structure constants in the ope of three ϕ_{rr} fields

$$b_{ij}^k = \lim_{\epsilon \to 0} C^{(\alpha\alpha\alpha)(kk)}_{(ii)(jj)} \ , \qquad b_{ij}^1 = \lim_{\epsilon \to 0} C^{(\alpha\alpha\alpha)(11)}_{(ii)(jj)} = \delta_{ij} \ . \tag{3.24}$$

[†]In a non-unitary theory, L_0 need not be diagonalisable, in which case equation (3.20) may be replaced by $[L_m, \psi_i(z)] = z^{m+1}\,\mathrm{d}\psi_i/\mathrm{d}z + z^m(m+1)\,H_{ij}\,\psi_j(z)$, where H_{ij} is a nilpotent matrix. However, by our construction H_{ij} is identically zero.

It turns out that the finite dimensional associative, commutative algebra B defined by the constants b_{ij}^k allows a representation in terms of orthogonal projectors. That is, there are exactly $\dim B$ elements $P_i \in B$ such that $P_i P_j = \delta_{ij} P_i$. (The only way this could not be the case would be if there were some nilpotent elements in B, but our choice of normalisation $b_{ii}^1 = 1$ in eqn. (3.24) excludes this possibility.)

We have already seen in equation (3.22) that a field $\phi_{rr} \in \mathcal{H}^{(0)}$ acts on the fields in $\mathcal{H}^{(k)}$ in different ways for $x < 0$ and $x > 0$. As a result, we have two commuting actions of B on the Hilbert space $\widehat{\mathcal{H}}_{\alpha\alpha}$, or by an abuse of notation, we have two commuting actions of two copies B^R and B^L of the algebra, defined by

$$
\begin{aligned}
\phi_{rr} \in B^R &: \Psi(0) \longmapsto \Psi(0) \cdot \phi_{rr}(-1) , \\
\phi_{rr} \in B^L &: \Psi(0) \longmapsto \phi_{rr}(1) \cdot \Psi(0) .
\end{aligned}
\tag{3.25}
$$

Given the decomposition (3.13),

$$
\lim_{c \to 1} Z_{\alpha\alpha} = \sum_{\{\alpha_i\},\{\alpha_j\}} \widehat{Z}_{(\widehat{\alpha}_i)(\widehat{\alpha}_j)} ,
$$

it is natural to assume that the projectors P_i^L and P_j^R project onto the subspace $\widehat{\mathcal{H}}_{\alpha_i \alpha_j}$ of $\widehat{\mathcal{H}}_{\alpha\alpha}$ corresponding to the fundamental boundary conditions $(\widehat{\alpha}_i)$ and $(\widehat{\alpha}_j)$ on either side of the origin, and leads directly to our conjecture 2. Again, we have not been able to prove this conjecture, but have checked it quite extensively, and present the results for the case $\alpha = (2,2)$ in section 3.3 and for the $(2,p)$ and $(3,p)$ boundaries in sections A.3 and A.4.

3.3 Example: the $c \to 1$ limit of the (2,2) boundary

In this example we will investigate the spaces $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ of the $(2,2)$ boundary, and show how the fields can be expressed through fields in the superposition of the $(\widehat{1})$ and $(\widehat{3})$ b.c.'s.

For c sufficiently close (but not equal) to one the Hilbert space of the (2,2) boundary decomposes as

$$
\mathcal{H}_{(22)(22)} = R_{(1,1)} \oplus R_{(3,3)} \oplus R_{(1,3)} \oplus R_{(3,1)} ,
\tag{3.26}
$$

For $c < 1$ there are four boundary primary fields,

$$
1 \equiv \phi_{(1,1)}^{((2,2)(2,2))} , \quad \phi \equiv \phi_{(3,3)}^{((2,2)(2,2))} , \quad \psi \equiv \phi_{(1,3)}^{((2,2)(2,2))} , \quad \bar{\psi} \equiv \phi_{(3,1)}^{((2,2)(2,2))} ,
\tag{3.27}
$$

and their weights are

$$
h_{1,1} = 0 , \quad h_{3,3} = \frac{2\epsilon^2}{1-\epsilon} , \quad h_{1,3} = 1 - 2\epsilon , \quad h_{3,1} = \frac{1+\epsilon}{1-\epsilon} .
\tag{3.28}
$$

In the limit $c \to 1$, the subspace $\mathcal{H}^{(0)}$ is spanned by the two primary fields

$$
\mathcal{H}^{(0)} = \{1, \phi\} .
\tag{3.29}
$$

The only nontrivial ope amongst these fields is given in (A.23)

$$
\phi(x) \, \phi(y) = 1 + \tfrac{2}{\sqrt{3}} \phi(y) ,
\tag{3.30}
$$

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and this defines our algebra B, with two generators 1 (serving as identity) and ϕ with relation (3.30). One identifies the two projectors as

$$
\begin{aligned}
P_a &= \tfrac{1}{4}\left(1 + \sqrt{3}\,\phi\right) \\
P_b &= \tfrac{1}{4}\left(3 - \sqrt{3}\,\phi\right)
\end{aligned}
\quad , \quad
\begin{aligned}
1 &= P_a + P_b \\
\phi &= \sqrt{3}P_a - \tfrac{1}{\sqrt{3}}P_b
\end{aligned}
\quad . \tag{3.31}
$$

To decide which of P_a and P_b is the projector onto the $\widehat{(1)}$ b.c. and which onto the $\widehat{(3)}$ b.c., we need the action of these projectors on the weight one fields.

In the limit $c \to 1$, the space $\mathcal{H}^{(1)}$ is generated by three primary fields of which two are the primary fields ψ and $\bar{\psi}$ corresponding to the spaces $R_{(13)}$ and $R_{(31)}$ in the decomposition (3.26), and the third is the field d_3 introduced in section 3.1.1,

$$
d_3 \equiv \lim_{\epsilon \to 0} \frac{\sqrt{1-\epsilon}}{2\epsilon} \frac{\mathrm{d}\phi}{\mathrm{d}z} \ . \tag{3.32}
$$

The opes of the field ϕ with the fields of weight one are given in (A.23), from which we can read off the actions of $\phi(1) = \phi^L$ and $\phi(-1) = \phi^R$ on the states $(|\psi\rangle, |\bar{\psi}\rangle, |d_3\rangle)$ and assemble them into matrices:

$$
\phi^L = \begin{pmatrix} 0 & \sqrt{\tfrac{1}{3}} & -\sqrt{\tfrac{2}{3}} \\ \sqrt{\tfrac{1}{3}} & 0 & -\sqrt{\tfrac{2}{3}} \\ -\sqrt{\tfrac{2}{3}} & -\sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{1}{3}} \end{pmatrix} , \quad
\phi^R = \begin{pmatrix} 0 & \sqrt{\tfrac{1}{3}} & \sqrt{\tfrac{2}{3}} \\ \sqrt{\tfrac{1}{3}} & 0 & \sqrt{\tfrac{2}{3}} \\ \sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{1}{3}} \end{pmatrix} . \tag{3.33}
$$

We can now determine the action of the four projectors P_a^R, P_a^L, P_b^R, P_b^L on $\mathcal{H}^{(1)}$. Of particular interest is the product of a left and a right projector as it gives the decomposition (3.4). The result is summed up in the following table:

projectors	image in $\mathcal{H}^{(1)}$	interpretation on $\widehat{(1)} \oplus \widehat{(3)}$ boundary
$P_a^L P_a^R$	0	no weight 1 field on $\underline{\widehat{1}}\,,\underline{\widehat{1}}$
$P_a^L P_b^R$	$\lambda \cdot (1,\ 1, -\sqrt{2})$	$\underline{\widehat{1}}\ \underline{\widehat{\varphi}_3^{(13)}}\ \underline{\widehat{3}}$ boundary changing field
$P_b^L P_a^R$	$\lambda \cdot (1,\ 1,\ \sqrt{2})$	$\underline{\widehat{3}}\ \underline{\widehat{\varphi}_3^{(31)}}\ \underline{\widehat{1}}$ boundary changing field
$P_b^L P_b^R$	$\lambda \cdot (1, -1, 0)$	$\underline{\widehat{3}}\ \underline{\widehat{\varphi}_3^{(33)}}\ \underline{\widehat{3}}$ boundary field

Table 1: The images of the projectors on the $c \to 1$ limit of the $(2,2)$ boundary

We see that the interpretation of P_i as projectors is consistent with the interpretation of the $c \to 1$ limit of the $(2,2)$ boundary as $\widehat{(1)} \oplus \widehat{(3)}$. On this superposition, there are six primary boundary fields: there are two of weight zero (the identity fields $\mathbf{1}^{(1)}$, $\mathbf{1}^{(3)}$ on each boundary condition), three of weight one (the boundary field $\widehat{\varphi}_3^{(33)}$ and the boundary changing fields $\widehat{\varphi}_3^{(13)}$, $\widehat{\varphi}_3^{(31)}$) and one of weight 4 (the boundary field $\widehat{\varphi}_5^{(33)}$). We shall only pay attention to the fields of weight zero and one. Combining equation (3.31) and table 1, we expect the relation

between the $c = 1$ fields and the $c \to 1$ limit of $(2,2)$ boundary fields to be

$$\mathbf{1}^{(1)} = \tfrac{1}{4}(1 + \sqrt{3}\phi) , \quad \mathbf{1}^{(3)} = \tfrac{1}{4}(3 - \sqrt{3}\phi) ,$$

$$\widehat{\varphi}_3^{(33)} = \lambda_1 \left(\psi - \bar{\psi}\right) , \quad \widehat{\varphi}_3^{(13)} = \lambda_2 \left(\psi + \bar{\psi} - \sqrt{2}d_3\right) , \quad \widehat{\varphi}_3^{(31)} = \lambda_3 \left(\psi + \bar{\psi} + \sqrt{2}d_3\right) . \tag{3.34}$$

for some values of $\lambda_1, \lambda_2, \lambda_3$.

We have already checked that the opes of the weight zero fields, and of weight zero fields with weight one fields are in agreement with this assignment. Now we check that the opes of the weight one fields on the $c \to 1$ limit of the $(2,2)$–boundary, summed up in equation (A.24), reproduces those on the boundary condition $(\widehat{1})\oplus(\widehat{3})$ of the $c=1$ model. The opes of the $c=1$ fields can be obtained from the structure constants given in the appendix. The non-zero opes are (all with $x > y$)

$$\mathbf{1}^{(a)} \quad \mathbf{1}^{(b)} \;=\; \delta_{a,b}\, \mathbf{1}^{(a)} ,$$

$$\mathbf{1}^{(a)} \quad \widehat{\varphi}_3^{(bc)}(y) \;=\; \delta_{a,b}\, \widehat{\varphi}_3^{(bc)}(y) ,$$

$$\widehat{\varphi}_3^{(bc)}(x) \quad \mathbf{1}^{(a)} \;=\; \delta_{a,c}\, \widehat{\varphi}_3^{(bc)}(x) ,$$

$$\widehat{\varphi}_3^{(13)}(x) \;\; \widehat{\varphi}_3^{(31)}(y) \;=\; \tfrac{1}{(x-y)^2}\, \mathbf{1}^{(1)} \;\; + \; O(1) ,$$

$$\widehat{\varphi}_3^{(13)}(x) \;\; \widehat{\varphi}_3^{(33)}(y) \;=\; \tfrac{2}{(x-y)}\, \widehat{\varphi}_3^{(13)}(y) \; + \; O(1) , \tag{3.35}$$

$$\widehat{\varphi}_3^{(31)}(x) \;\; \widehat{\varphi}_3^{(13)}(y) \;=\; \tfrac{1}{3}\tfrac{1}{(x-y)^2}\, \mathbf{1}^{(3)} \;\; + \; \tfrac{2}{3}\tfrac{1}{(x-y)}\, \widehat{\varphi}_3^{(33)}(y) \; + \; O(1) ,$$

$$\widehat{\varphi}_3^{(33)}(x) \;\; \widehat{\varphi}_3^{(31)}(y) \;=\; \tfrac{2}{(x-y)}\, \widehat{\varphi}_3^{(31)}(y) \; + \; O(1) ,$$

$$\widehat{\varphi}_3^{(33)}(x) \;\; \widehat{\varphi}_3^{(33)}(y) \;=\; \tfrac{1}{(x-y)^2}\, \mathbf{1}^{(3)} \;\; + \; \tfrac{1}{(x-y)}\, \widehat{\varphi}_3^{(33)}(y) \;\; + \; O(1) .$$

The opes such as $\widehat{\varphi}_3^{(13)}(x)\,\widehat{\varphi}_3^{(13)}(y)$, $\widehat{\varphi}_3^{(31)}(x)\,\widehat{\varphi}_3^{(31)}(y)$, etc, have to be zero because the boundary conditions do not match up. Substituting (3.34) one verifies that these opes vanish for any choice of $\lambda_1, \lambda_2, \lambda_3$, and that the five nontrivial opes in (3.35) are correctly reproduced for $\lambda_1 = -\sqrt{3/8}$ and $\lambda_2\lambda_3 = 1/16$.

This does not, of course, represent a complete proof of our conjecture, since we have not treated the weight four primary boundary field, nor any fields of weight greater than one, and we do not in any case have an independent proof that the $c=1$ b.c.'s $(\widehat{r})$ arise as the boundary conditions of any particular $c=1$ bulk theory, but we regard it as very strong evidence for conjectures 1 and 2.

4   The perturbations by ϕ_{rr}

We can now examine the ϕ_{rr} perturbations of the minimal models in the light of these results.

The first point is that the situation at $c=1$ is entirely clear. Consider the $c \to 1$ limit of the field ϕ_{rr} on the (a, a') boundary condition. This can be expanded in terms of the projectors P_i onto the fundamental components on the decomposition (3.3)

$$\phi_{rr} = \sum \mu_i^{(r)} P_i . \tag{4.1}$$

We consider the perturbation of the theory on a strip of width R with boundary condition $\alpha = (a, a')$ on the right edge and $\beta = (b, b')$ on the left by the addition of the field $\phi_{rr}^{(\alpha\alpha)}$ on the right edge,

$$S = S_0 \ + \ \lambda \int \phi_{rr}^{(\alpha\alpha)}(x)\, \mathrm{d}x . \tag{4.2}$$

This can be reformulated in terms of a perturbed Hamiltonian on the UHP

$$H \ = \ \left(\frac{\pi}{R}\right) \left[L_0 \ - \ \frac{c}{24} \ + \ \lambda\, \phi_{rr}(1) \right] \ = \ \left(\frac{\pi}{R}\right) \left[L_0 \ - \ \frac{c}{24} \ + \ \lambda \sum \mu_i^{(r)} P_i^R \right] . \tag{4.3}$$

It is clear that the effect of the addition of the perturbation is just to add an amount $\lambda\,\mu_i^{(r)}$ to the energy of a state in the fundamental component i on the right boundary. As $|\lambda| \to \infty$, only the sector(s) with minimal $(\lambda\mu_i^{(r)})$ will survive (with all the other sectors decoupling) i.e. for $\lambda > 0$, the right boundary flows to the system with boundary condition $\oplus(\widehat{c})$ with $\mu_c^{(r)}$ minimal, and for $\lambda < 0$, the right boundary flows to the system with boundary condition $\oplus(\widehat{c})$ with $\mu_c^{(r)}$ maximal.

To return to the example we have treated in depth, consider the model with boundary condition $(11) \equiv (\widehat{1})$ on the left edge and $(22) \equiv (\widehat{1}) \oplus (\widehat{3})$ on the right edge. There is a single non-trivial boundary field ϕ_{33} on the (22) boundary, which can be expressed in terms of the projectors $P_{(\widehat{1})}$ and $P_{(\widehat{3})}$ as

$$\phi_{33} = \sqrt{3}\, P_{(\widehat{1})} \ - \ \frac{1}{\sqrt{3}}\, P_{(\widehat{3})} . \tag{4.4}$$

So, for $\lambda > 0$ this system flows to the model on the strip with boundary conditions $(\widehat{1})$ on the left and $(\widehat{3})$ on the right, and for $\lambda < 0$ it flows to the model with boundary condition $(\widehat{1})$ on the left and $(\widehat{1})$ on the right.

There are several ways we can present this graphically. For a general perturbation, λ is not a dimensionless variable, so we define the dimensionless variables κ and r,

$$\kappa = \lambda\, R^y , \ \ r = |\kappa|^{1/y} , \quad y = 1 - h_{33} . \tag{4.5}$$

In figure 1a we can plot the eigenvalues of $(R/\pi)H$ against κ for fixed R, and in figures 1b and 1c we plot the gaps $R(E - E_0)/\pi$ above the ground state energy against $\log|r|$ for λ positive and negative respectively. We show these for later comparison with the equivalent plots for $c < 1$.

We should comment that the apparent lack of smoothness in figure 1b at $\kappa = \sqrt{3}/4$ is due to the fact we are plotting the scaled energy *gaps*, and that the first excited state crosses the ground state at that value of κ.
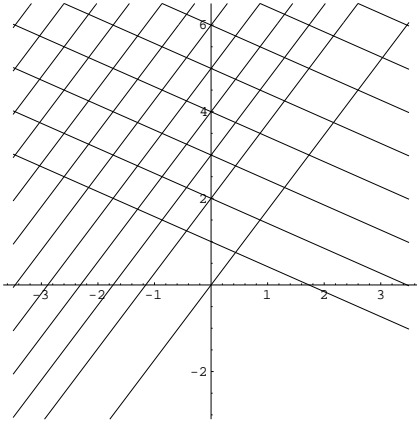
Figure 1a: $(22) \pm \lambda\phi_{33}$.
The eigenvalues of $(R/\pi)H(\kappa)$
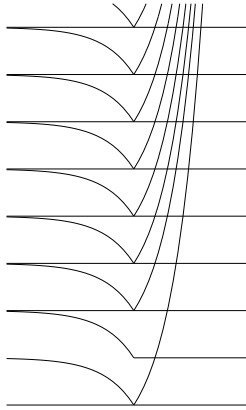plotted against κ.

Figure 1b: $(22) + \lambda\phi_{33}$.
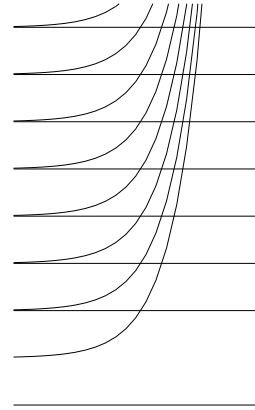Energy gaps plotted
against $\log|r|$.

Figure 1c: $(22) - \lambda\phi_{33}$.
Energy gaps plotted
against $\log|r|$.

4.1 The minimal models with $c < 1$

For $c < 1$ but ϵ still small, we expect that this picture will only change slightly, and in particular the IR end points of these flows should agree with the results at $c = 1$. This leaves some ambiguity, however, since several different boundary conditions for $c < 1$ may have the same limit at $c = 1$. In the particular case of the (22) boundary condition perturbed by ϕ_{33}, the IR limits at $c = 1$ are $\widehat{(1)}$ and $\widehat{(3)}$ for λ negative and positive respectively. There is a single boundary condition which has as its limit $\widehat{(1)}$, namely (11); however both (13) and (31) have as their limits $\widehat{(3)}$. We must find a way to decide which is the correct IR endpoint for $c < 1$.

One method might be to use conformal perturbation theory, but since the perturbation is UV–finite and IR–divergent, the conformal perturbation theory tells us nothing about the IR end point, as was also the case for the Lee-Yang model studied in [11, 12].

The only other method open to us at the moment is the Truncated Conformal Space Approach (TCSA) [11]. In figures 2a–4c we show the equivalent plots for the perturbation of the strip with boundary conditions (11) and (22) by ϕ_{33} for the minimal models $M_{10,11}$, $M_{6,7}$ and $M_{4,5}$, all calculated using TCSA.

We see that the pattern in $M(10, 11)$ is extremely similar to that at $c = 1$, but that the lines no longer cross. This is the typical behaviour of a non-integrable flow – the folklore being that one can only expect line crossings if there are conserved quantities present which forbid mixing of states. These 'gaps' open up further as ϵ grows in $M(6, 7)$ and $M(4, 5)$, but we see that in $M(4, 5)$ the lines appear to cross again. This would indicate that this flow is again integrable with an infinite number of conserved quantities, and closer inspection shows that this is indeed likely to be the case as the symmetries of the Kac table mean that in the model $M(4, 5)$, $\phi_{33} \equiv \phi_{12}$, and ϕ_{12} is well known to be one of the generic integrable perturbations (along with $(1, 3)$ and $(1, 5)$ and the images under $(r, s) \leftrightarrow (s, r)$.) As shown in [16], the same arguments that are used to show the integrability of bulk perturbations can also be used to show the integrability of the analogous boundary perturbations. We have checked that there are also line-crossings in the perturbations of the boundary condition (33)
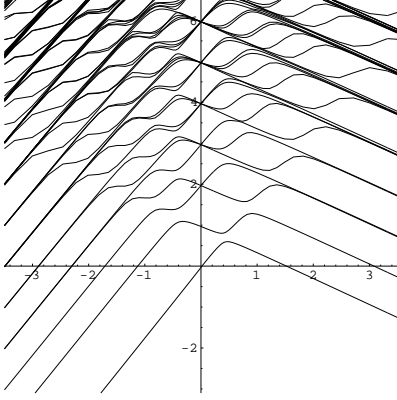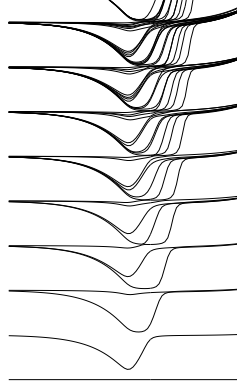
Figure 2a: $M_{10,11} : (22) \pm \lambda\phi_{33}$.

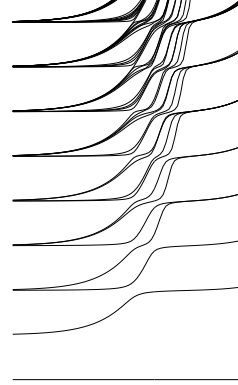Figure 2b:
$M_{10,11} : (22) + \lambda\phi_{33}$.

Figure 2c:
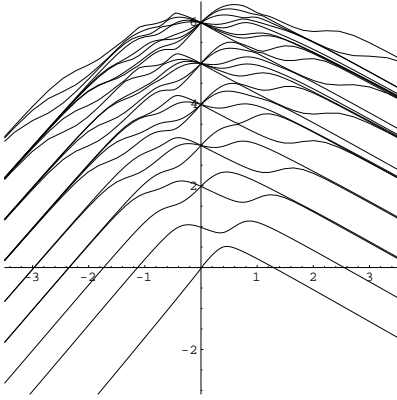$M_{10,11} : (22) - \lambda\phi_{33}$.

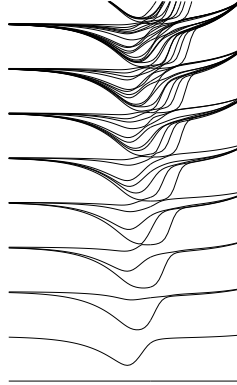Figure 3a: $M_{6,7} : (22) \pm \lambda\phi_{33}$.

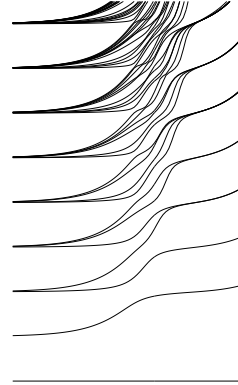Figure 3b:
$M_{6,7} : (22) + \lambda\phi_{33}$.
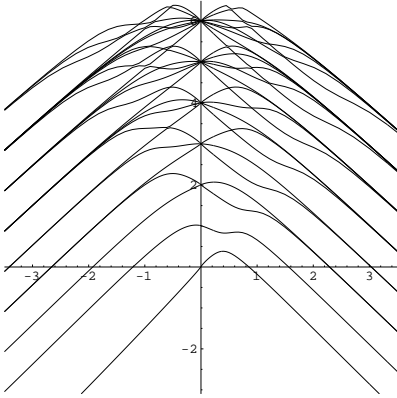
Figure 3c:
$M_{6,7} : (22) - \lambda\phi_{33}$.

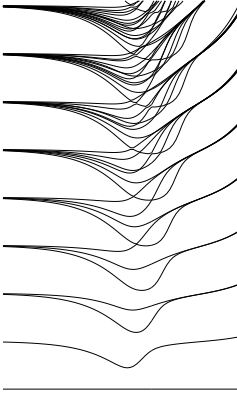Figure 4a: $M_{4,5} : (22) \pm \lambda\phi_{33}$.
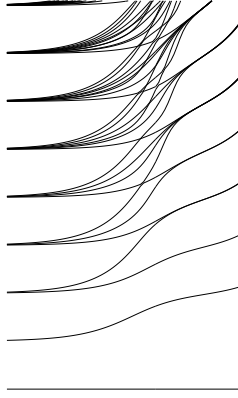
Figure 4b:
$M_{4,5} : (22) + \lambda\phi_{33}$.

Figure 4c:
$M_{4,5} : (22) - \lambda\phi_{33}$.

Figures 2a, 3a and 4a: The first 25 eigenvalues of $(R/\pi)H(\kappa)$ plotted against κ;
Figures 2b,2c, 3b,3c, 4b and 4c: Energy gaps $(R/\pi)(E_i - E_0)$ plotted against $\log|r|$

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by the field $\phi_{55} \equiv \phi_{12}$ in the model $M(6,7)$. and of the boundary condition (44) by the field $\phi_{77} \equiv \phi_{12}$ in the model $M(8,9)$.

We should also comment on the fact that the energy gaps do not tend to constants in the IR. This is due to truncation effects, and the effect decreases as the truncation level is increased.

The boundary condition (γ) at the IR fixed point for positive λ could be deduced from the TCSA plots in one of two ways. Firstly, we could use the asymptotic behaviour of the ground state energy to find the conformal weight of the ground state directly but this is problematic, as we shall discuss in the next section.

The easier method is to count the degeneracies of the IR spectrum, from which one can identify the partition function of the strip with boundary conditions (11) and (γ), since the partition function of a strip with boundary conditions (11) and $(\gamma) = \oplus (\gamma_i)$ is equal to the sum of characters of the representations γ_i:

$$Z_{(11)(\gamma)} = \sum_i \ \chi_{\gamma_i} \ . \tag{4.6}$$

In this case, we expect that (γ) will be a single representation (13) or (31), and so the partition function should be $\chi_{13}(q)$ or $\chi_{31}(q)$ respectively. We can identify the character by the counting of states, which will show the existence of null vectors at levels 3 and $(p-1)(p'-3)$ for boundary condition (13) and at levels 3 and $(p-3)(p'-1)$ for boundary condition (31). For the models $M_{10,11}$ and $M_{6,7}$, the second null vector is at too high a level to be calculated easily using TCSA, but for $M_{4,5}$ it should be at level 6 or 4 for the cases (13) and (31) respectively. From figure 4b we see that there is indeed a state missing at level 4, so we can positively identify this IR endpoint as the (31) representation.

We should make it clear that we cannot *prove* using TCSA that the IR endpoint of the flow $(22) - \phi_{33}$ is the b.c. $(1,1)$, since (quite apart from numerical errors) one can never be sure that the IR regime has been reached. At best we can say that for $\log |r| \sim 2$ (the right hand edge of the graphs 2b, 3b and 4b) the counting of states indicates that the flow is in the neighbourhood of the (31) b.c., and since that b.c. has no relevant perturbations it is reasonable to believe that it is the endpoint of the flow. Similarly, since the counting of states indicates that the flows $(22) + \phi_{33}$ enter the vicinity of the (31) b.c. which again has no relevant perturbations, this suggests that this is indeed the IR endpoint of this flow.

4.2 The scaling behaviour of the ground state energy

We expect the ground state energy calculated using TCSA to have three different scaling behaviours according to the value of R:

$$f(r) \equiv \frac{RE_0(R)}{\pi} \ \sim \ \begin{cases} (h_{\mathrm{UV}} - \frac{c}{24}) \ + \ c_1 \, r^y \ + \ \dots & R \text{ small,} \\ (h_{\mathrm{IR}} - \frac{c}{24}) \ + \ c_2 \, r \ + \ \dots & \text{The 'scaling region',} \\ c_3 \, r^y \ + \ \dots & R \text{ large, truncation errors} \\ & \text{dominate.} \end{cases} \tag{4.7}$$

(here c_2 is the IR boundary free-energy-per-unit-length, and $h_{\mathrm{UV/IR}}$ are the minimal weights of the UV and IR fixed points respectively.)

It is well known that the TCSA method cannot be applied easily to bulk massless flows as it is hard to reach the appropriate scaling region, for several reasons. Firstly, since the fixed

point may still have relevant perturbations, errors introduced by truncation can drive the flow away from the intended fixed point, and the corrections to the leading scaling behaviour can be large, decaying with powers of r, rather than exponentially. Secondly, high level states that are dropped by truncation can still contribute appreciably to the ground-state-energy. A recent exception to this rule is the double-Sine-Gordon model, where it has proven possible to obtain the flow to the Ising point using TCSA [2] by truncating at rather high levels and so decreasing the truncation errors, and by fine-tuning in two variables to hit the IR fixed point. To test for the onset of scaling, we can try to fit the ground state energy by a function of the form

$$f(r) \sim a + c r^b , \tag{4.8}$$

and estimate b by the function

$$b_{\text{est}}(r) = 1 + r \frac{\mathrm{d}}{\mathrm{d}r} \log \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right) . \tag{4.9}$$

In the scaling regime, we should obtain $b \approx 1$. Similarly, we can estimate a by using the expected scaling form (4.7) (i.e. taking $b = 1$ in (4.8)) to give

$$a_{\text{est}}(r) = -r^2 \frac{\mathrm{d}}{\mathrm{d}r} (f/r) . \tag{4.10}$$

In figure 5 we plot $a_{\text{est}}(r)$ and $b_{\text{est}}(r)$ against $\log |r|$ for the model $M_{6,7} - \lambda\phi_{33}$ calculated using TCSA with levels 6, 10 and 15, that is truncated to 26, 109 and 489 states respectively, and also an extrapolation of the data to infinite level. Also shown in these plots are the expected UV and IR behaviour. Although the finite level TCSA results do not show scaling – b_{est} does not tend to 1 and a_{est} does not tend to the expected constant – the extrapolated results are much better. However, even after extrapolation, we cannot really say that we have shown that the IR limit is indeed the one we expect.
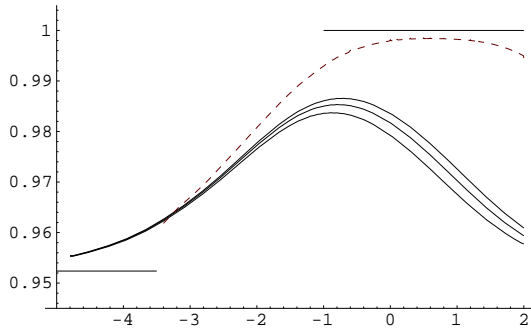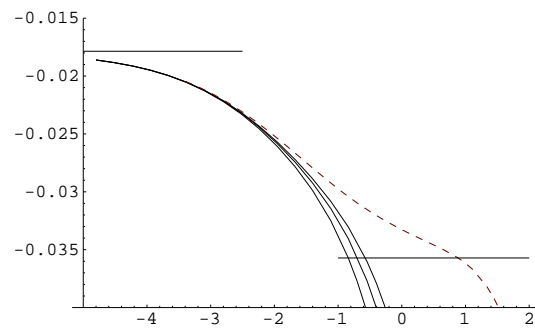

Figure 5a   Figure 5b

Plots of $b_{\text{est}}(r)$ (left) and $a_{\text{est}}(r)$ (right) from TCSA for $M_{6,7} : (22) - \lambda\phi_{33}$. The solid lines are from truncation to levels 6,10 and 15, and the dashed line an extrapolation to infinite level. Also shown are the expected UV and IR values (the horizontal lines).

We see from these plots that the dominant contribution to the errors comes from the truncation – the extrapolation of $b_{\text{est}}(r)$ in figure 5a suggests that scaling would set in for $\log |r| \gtrsim 0$, if the truncation errors could be removed. Unfortunately the corresponding results for the flow in the positive direction are not even as good, since the scaling region only appears to set in for $\log |r| > 2$.

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We can try to improve on these results by including in our fits some of the sub-leading contributions to (4.7)

$$f(r) \; = \; a \; + \; c\,r \; + \; \sum d_i\,r^{1-h_i} \; + \; \dots \;\; , \tag{4.11}$$

corresponding to the leading contributions from the quasi-primary irrelevant operators of weight h_i at the IR fixed point. These operators are $T(x)$ on the (11) b.c., and $T(x)$ and ϕ_{31} on the (31) b.c. While these do improve the fit to the IR behaviour, even including these extra corrections we do not see unambiguous signs that we are at the correct IR fixed point, and so do not present them here.

4.3 The nonunitary models

The question we must ask now, is whether this scheme we have outlined is also valid for perturbations of the non-unitary models $M(p,p')$ with $p' \neq p+1$.

For those models far from $t = 1$, the field ϕ_{rr} has weight greater than $1/2$. This means that in a proper field theory treatment the model needs to be regularised and renormalised and a large range of possible counter terms need to be considered. We certainly have no expectation that our results (which are based on the idea that ϕ_{rr} is close to a scalar field) will remain true in such a case. However, one might hope that for p' close to p this picture would still work. To answer this question it is important first to address the general dependence of the pattern of the flows on the boundary condition, the perturbing field and the central charge.

We first consider the generic situation with the parameter t irrational, and where the boundary condition (h) and perturbing field $\phi_{h'}$ are also generic, i.e. for which there are no null states in the representation R_h. In this case the spectrum depends smoothly on the parameters t, h and h' (n.b. we are not making any assertions about the existence or otherwise of a local field theory with these properties, only about the TCSA spectra as determined by the TCSA matrix elements). The only singularities occur when t and h are such that there are null states in R_h, in which the spectrum is given by the generic pattern but with the omission of certain complete lines corresponding to the decoupling of the null states[‡].If the null states that are decoupled are above the truncation level, then no difference will be seen on the TCSA plots.

In this section we have mostly focussed on the flows $(22) \pm \phi_{33}$, for which there are always null states in the representation $R_{(2,2)}$, starting at level 4. There are extra null states for rational values of the parameter $t = p/p'$, starting at level $(p-2)(p'-2)$. In figures 2a–4c, we have shown states up to level 8, so that the pattern is generic (for these flows) apart from the cases

$$t \; \in \; \{\, \frac{3}{4}, \;\; \frac{3}{5}, \;\; \frac{3}{7}, \;\; \frac{3}{8}, \;\; \frac{4}{5} \,\} \;. \tag{4.12}$$

The last case we have looked at $M(4,5)$, is one of these special values, so that we should also look at a 'neighbouring' flow to see the generic situation. In figures 6a–b we show the plots for $t = 0.8$ (truncated to level 14) in bold, with the extra lines for $t = 0.8002$ superposed as dashed lines. We also indicate (with a vertical dotted line) how far we think the qualitative features of this graph can be trusted.

[‡]A similar phenomenon occurs for the bulk perturbations by the field ϕ_{13}, where the minimal model spectra are a subset of the spectra of the folded sine-Gordon model [3]
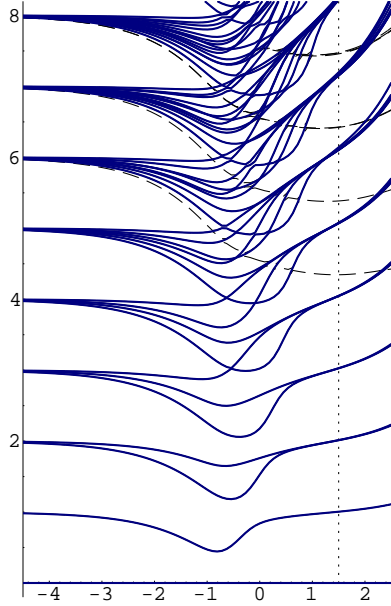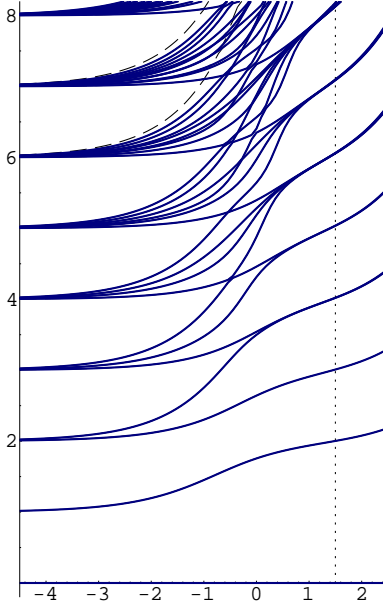
Figure 6a   Figure 6b
Energy gaps plotted against $\log|r|$ for $M_{t=0.8002} : (22) + \lambda\phi_{33}$ (left) and
$M_{t=0.8002} : (22) - \lambda\phi_{33}$ (right) from TCSA to level 14 (369 states).

It is important to work out how far these graphs can be trusted, since the first 'extra' line in figure 6a appears to descend from level 6 (the first extra null state in $R_{(2,2)}$ for $t = 4/5$) first to level 4 (the first extra null state in $R_{(3,1)}$ for $t = 4/5$) and further. If this level really dropped below energy 4 then we would have trouble identifying the spectrum as that of the boundary condition $(3, 1)$. One way to judge how far these graphs can be trusted is to see how the pattern changes with the truncation level. In figure 7 we plot the normalised energy gaps $(E_i(r) - E_0(r))/(E_1(r) - E_0(r))$ for $\log|r| > 0$ in the case $t = 0.8$ for truncation 12, and on top of this we plot the first 'extra' line for $t = 0.8002$ for truncation levels 8, 10, 12 and 14. (It is important to note that the lines crossings in $M(4, 5)$ are absent for $M(4001, 5000)$, but that the gaps between the lines are so small that one can easily identify the 'extra lines' that we plot here.) It is clear that the spectrum is not really stable for $\log|r| > 2$, and that the crossing of the level 4 by the extra line is never part of the stable spectrum – hence one could easily believe that the first extra line will really join the other lines at energy level 4 after truncation effects are removed.

This discussion suggests that for $t > 4/5$ the endpoint of the $(22)\pm\phi_{33}$ flows are unchanged from those of the unitary models, and furthermore it appears that the spectrum stays real for all $4/5 \leq t < 1$, although we do not have any arguments to support this. For $t < 4/5$, we find that the spectrum develops large imaginary parts, and this makes the identification of the IR fixed point much harder, and so we shall not attempt to say anything more about this regime.
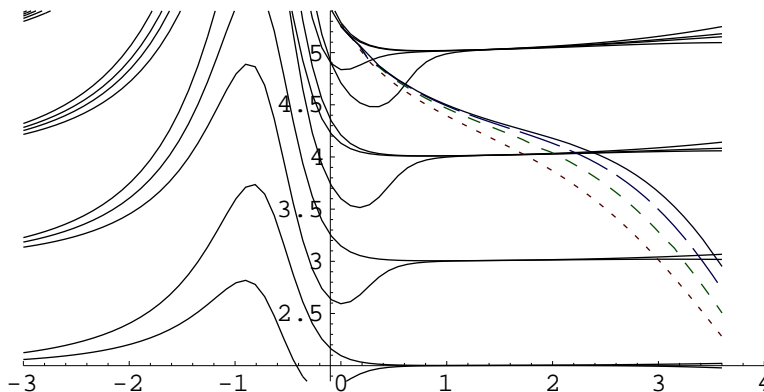
Figure 7: $(22) + \lambda\phi_{33}$.
The normalised energy gaps for $M(4,5)$ plotted against $\log|r|$, with the first 'extra' line for $M(4001, 5000)$ superposed. Truncation levels are 8 (dotted), 10 (short dashed), 12 (long dashed) and 14 (solid).

5   Conclusions

We have shown how one may take the $c \to 1$ limit of the A–type minimal models with $c<1$ with a boundary, and that in this limit the boundary condition of type (rr') splits into a superposition of $\min(r, r')$ fundamental boundary conditions.

This leads to a simple heuristic picture for the perturbations of the $c<1$ minimal models by the boundary fields ϕ_{rr} of 'low' conformal weight. We have checked that this picture appears to be correct in various unitary models through use of the TCSA method. We have argued that the pattern of flows changes smoothly with central charge (modulo the omission of lines corresponding to null states) and that for the perturbation by ϕ_{33} the IR endpoints of the nonunitary models with $t \geq 4/5$, that is $c \geq 7/10$, appear to be the same as those of the unitary models.

We have also found good evidence in the models $M(4,5)$, $M(6,7)$ and $M(8,9)$ that the perturbation of the boundary condition $(2p, 2p+1)$ by the field $\phi_{2p-1,2p-1} \equiv \phi_{12}$ is integrable (as one would expect on general grounds [16]). This clearly deserves further investigation, as they may be amenable to an exact analysis through non-linear integral techniques.

Questions which we also plan to consider in the future are whether there are any new features in the D-type models, and whether we can identify a bulk model for which the $c=1$ boundary conditions $(\widehat{r})$ we have found are the natural boundary conditions. This last point will be addressed in [25].

Finally, in a recent paper [27], Zamolodchikov and Zamolodchikov considered Liouville theory with $c \geq 25$, found that the boundary conditions are naturally labelled (m, n), and noted that the subset $(1, n)$ has a special role. It would be interesting to see if there is any relation to the $c=1$ boundary conditions presented here.

Appendices

A.1   A–series boundary structure constants

In [24] it was argued that the boundary structure constants of A–series minimal models are given by the Fusion-matrix F, which describes the transformation behaviour of conformal blocks. The result is most simply put as

$$\tilde{C}_{ij}^{(abc)k} = \mathsf{F}_{bk}\begin{bmatrix} a & c \\ i & j \end{bmatrix} . \tag{A.1}$$

However, this solution has to be rescaled to match the normalisation in (2.11). To investigate the $\epsilon \to 0$ limit of this expression it is convenient to have an explicit expression for the structure constants. This is given in the next section.

A.1.1   Explicit minimal model F-matrix

Consider the minimal model $M(p,p')$. Let $t=p/p'$ and $d_{rs}=r-st$. We want to find the matrix connecting the conformal blocks occurring in the $x\to 0$ and $x\to 1$ expansion of the chiral correlator $\langle\, \phi_I\,|\phi_J(1)\phi_K(x)|\,\phi_L\,\rangle$. The indices are given by Kac-labels $I=(i,i')$, $J=(j,j')$, etc. Let correspondingly $d_I=i-i't$, $d_J=j-j't$, etc.
From [13](A.35) we find:

$$b_{xy}(\alpha,\beta;\rho) \;=\; \prod_{g=1}^{y} \frac{\Gamma(g\rho)\Gamma(\alpha+g\rho)\Gamma(\beta+g\rho)}{\Gamma(\rho)\Gamma(\alpha+\beta-2x+(y+g)\rho)} \;, \tag{A.2}$$

$$m_{xy}(\alpha,\beta) \;=\; t^{2xy}\prod_{g=1}^{x}\prod_{h=1}^{y}\Big\{(ht-g)(\alpha+ht-g)(\beta+ht-g)(\alpha+\beta+(y+h)t-(x+g))\Big\}^{-1} \tag{A.3}$$

$$j(x,y;\alpha,\beta) \;=\; m_{xy}(\alpha,\beta)\cdot b_{yx}(-\tfrac{1}{t}\alpha,-\tfrac{1}{t}\beta;\tfrac{1}{t})\cdot b_{xy}(\alpha,\beta;t) \;. \tag{A.4}$$

From [15](3.5) we find:

$$
a(s;x,y;\alpha,\beta,\gamma,\delta;\rho) \;=\; \sum_{h=\max(x,y)}^{\min(s,x+y-1)} \frac{\displaystyle\prod_{g=1}^{s-h}\sin\pi(\delta+(x-1+g)\rho)\prod_{g=1}^{h-y}\sin\pi(-\alpha+(s-x+g)\rho)}{\displaystyle\prod_{g=1}^{s-y}\sin\pi(-\alpha+\delta+(s-y+g)\rho)}
$$

$$
\times\,\frac{\displaystyle\prod_{g=1}^{y-1-(h-x)}\sin\pi(\beta+(s-x+g)\rho)\prod_{g=1}^{h-x}\sin\pi(\gamma+(x-1+g)\rho)}{\displaystyle\prod_{g=1}^{y-1}\sin\pi(\beta+\gamma+(y-1+g)\rho)}
$$

$$
\times\,\prod_{g=1}^{h-x}\frac{\sin\pi((x+y-h-1+g)\rho)}{\sin\pi(g\rho)}\prod_{g=1}^{s-h}\frac{\sin\pi((h-y+g)\rho)}{\sin\pi(g\rho)}
\tag{A.5}
$$

Putting together [13](4.1) and [15](3.1) we find:

$$
\begin{aligned}
&\mathsf{F}_{PQ}\begin{bmatrix} J & K \\ I & L \end{bmatrix} \\
&= \frac{j(\frac{1}{2}(\ell-i-1+q),\frac{1}{2}(\ell'-i'-1+q');-d_I,d_L)}{j(\frac{1}{2}(j-i-1+p),\frac{1}{2}(j'-i'-1+p');-d_I,d_J)}\; \frac{j(\frac{1}{2}(j+k-1-q),\frac{1}{2}(j'+k'-1-q');d_J,d_K)}{j(\frac{1}{2}(k+\ell-1-p),\frac{1}{2}(k'+\ell'-1-p');d_K,d_L)} \\
&\quad\times\; a(\tfrac{1}{2}(-i+j+k+\ell);\tfrac{1}{2}(k+\ell+1-p),\tfrac{1}{2}(j+k+1-q);-\tfrac{1}{t}d_I,-\tfrac{1}{t}d_J,-\tfrac{1}{t}d_K,-\tfrac{1}{t}d_L;\tfrac{1}{t}) \\
&\quad\times\; a(\tfrac{1}{2}(-i'+j'+k'+\ell');\tfrac{1}{2}(k'+\ell'+1-p'),\tfrac{1}{2}(j'+k'+1-q');d_I,d_J,d_K,d_L;t)
\end{aligned}
\tag{A.6}
$$

A.1.2   The $c \to 1$ limit of the boundary structure constants

We shall take all boundary fields on a single b.c. (α) to be canonically normalised,

$$
\phi_i^{(\alpha\alpha)}(x)\,\phi_j^{(\alpha\alpha)}(y) = \frac{\delta_{ij}}{(x-y)^{2h_i}} + \sum_k C_{ij}^{(\alpha\alpha\alpha)k}\,(x-y)^{h_k-h_i-h_j}\,\phi_k^{(\alpha\alpha)}(y) + \dots \quad x>y\,,\quad \text{(A.7)}
$$

so that we only need give the structure constants

$$
C_{ijk}^{(\alpha)} \equiv C_{ij}^{(\alpha\alpha\alpha)k}\,,
\tag{A.8}
$$

which are cyclically symmetric (n.b. in the A–model with $c<1$ these are also completely symmetric, but that is not the case for the D–type model). However, when we consider boundary-condition changing operators, one cannot set both $C_{ii}^{(aba)1}$ and $C_{ii}^{(bab)1}$ to one due to the (normalisation independent) condition [20, 24]

$$
C_{ii}^{(aba)1}\,S_1^a = C_{ii}^{(bab)1}\,S_1^b\,,
\tag{A.9}
$$

Instead we choose an ordering on the boundary conditions and set

$$
C_{ii}^{(aba)1} = \begin{cases} 1\,, & a<b \\ S_1^b/S_1^a\,, & a>b \end{cases}
\tag{A.10}
$$

With these normalisations, one observes that the structure constants have a well-defined limit, independent of the precise choice of sequence or even if the minimal models in the sequence are unitary or not.

It turns out to be possible to find quite concise explicit formulae for the limits of the boundary structure constants involving only the boundary conditions ($\widehat{a}$) and the corresponding fields $\widehat{\varphi}_r^{(ab)} = \lim_{c\to 1} \phi_{(1r)}^{(1a)(1b)}$, which we present in the next section.

A.1.3   Limit of F for (1,p)-representations

Define ϵ by $t = 1 - \epsilon$ and let $j,k,\ell,p,q \in \mathbb{N}$ fulfill the conditions

$$
\begin{array}{llll}
|i-j| < p < i+j & |k-\ell| < p < k+\ell & \text{where } i+j+p \text{ and } k+\ell+p \text{ are odd,} \\
|i-l| < q < i+\ell & |j-k| < q < j+k & \text{where } i+\ell+q \text{ and } j+k+q \text{ are odd.}
\end{array}
\tag{A.11}
$$

We shall denote the limit of the F-matrix by

$$\widehat{\mathsf{F}}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = \lim_{\epsilon \to 0} \mathsf{F}_{(1,p)(1,q)}\begin{bmatrix} (1,j) & (1,k) \\ (1,i) & (1,\ell) \end{bmatrix} . \tag{A.12}$$

Using the explicit expression for the minimal model F-matrix we find that the limit $\epsilon \to 0$ is finite if condition (A.11) is fulfilled. Let $s = (-i+j+k+\ell)/2$, $x = (k+\ell+1-p)/2$, $y = (j+k+1-q)/2$. Then

$$\widehat{\mathsf{F}}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = (-1)^{(s+k)(s+x+y+1)} \frac{(k+\ell-x-1)!}{(k+\ell-2x)!} \prod_{g=1}^{s-y} \frac{g!\,(i+g-2)!}{(i+s-y-\ell+g)!\,(\ell-g)!}$$
$$\times \prod_{g=1}^{s-x} \frac{(i+s-x-j+g-1)!\,(j-g)!}{(g-1)!\,(i+g-2)!} \prod_{g=1}^{x-1} \frac{(\ell-x+g)!\,(k-x+g)!}{(g-1)!\,(k+\ell-2x+g+1)!} \prod_{g=1}^{y-1} \frac{g!\,(j+k-2y+g-1)!}{(j-y+g)!\,(k-y+g)!}$$
$$\times \sum_{h=\max(x,y)}^{\min(s,x+y-1)} \frac{\prod_{g=1}^{s-h}(x-\ell-1+g)\prod_{g=1}^{x+y-1-h}(x+j-s-g)\prod_{g=1}^{h-x}(k-x+1-g)\prod_{g=1}^{h-y}(i+s-x+g)}{(h-x)!\,(h-y)!\,(x+y-h-1)!\,(s-h)!} . \tag{A.13}$$

For the indices in the range (A.11) the arguments of the factorials are always non-negative. To normalise the structure constants we need the F-matrix elements corresponding to the two-point functions of boundary fields. Let $n = (a-b+i+1)/2$, then

$$\widehat{\mathsf{F}}_{b1}\begin{bmatrix} a & a \\ i & i \end{bmatrix} = \frac{b}{a-n+i} \frac{(a-n)!\,(i-n)!\,(i+a-n-1)!}{(n-1)!\,(i-1)!\,(a-1)!\,(b-1)!} \left\{ \prod_{g=1}^{n-1} \frac{(g+a-n)!\,(g+i-n)!}{(g-1)!\,(g-1+b)!} \right\}^2 . \tag{A.14}$$

One can verify that (A.14) is positive provided the indices are in their allowed ranges (A.11). Since each sequence of $\widehat{\mathsf{F}}$'s has a well-defined limit (A.12), taking the limit commutes with addition and multiplication. It follows that the $\widehat{\mathsf{F}}$'s fulfill the pentagon identity. Define the constants

$$a \le b : \quad A_i^{ab} = \left(\widehat{\mathsf{F}}_{b1}\begin{bmatrix} a & a \\ i & i \end{bmatrix} \right)^{1/2} > 0 \quad \text{and} \quad a > b : \quad A_i^{ab} = \left(\widehat{\mathsf{F}}_{a1}\begin{bmatrix} b & b \\ i & i \end{bmatrix} \right)^{1/2} > 0 . \tag{A.15}$$

Then the structure constants in the $c = 1$ theory are given by

$$\widehat{C}_{ij}^{(abc)k} = \frac{A_k^{ac}}{A_i^{ab} A_j^{bc}} \widehat{\mathsf{F}}_{bk}\begin{bmatrix} a & c \\ i & j \end{bmatrix} . \tag{A.16}$$

The normalisation has been chosen such that all structure constants are real and $\widehat{C}_{ii}^{(aaa)1}=1$. For boundary changing operators we have $\widehat{C}_{ii}^{(aba)1}=1$ if $a < b$ and $\widehat{C}_{ii}^{(aba)1}=b/a$ if $a > b$. Since the $\widehat{\mathsf{F}}$'s fulfill the pentagon identity, the structure constants (A.16) solve the boundary sewing constraint given in [20].

A.2 The $c \to 1$ limit of the (2,2)-boundary

In the limit $c \to 1$, there are two primary boundary fields of weight zero,

$$1 = \phi_{11} \ , \quad \phi = \phi_{33} \ , \tag{A.17}$$

and three fields of weight one

$$\psi = \phi_{13} \ , \quad \bar{\psi} = \phi_{31} \ , \quad d_3 = \lim_{\epsilon \to 0} \frac{\sqrt{1-\epsilon}}{2\epsilon} \phi'_{33} \ , \tag{A.18}$$

where $'$ denotes the derivative along the boundary. To work out the opes of these fields we need the structure constants for $c < 1$ to order $O(\epsilon)$. Since we normalise the fields on the $(2,2)$ boundary, we only need give the cyclically symmetric structure constants (A.8):

$$
\begin{aligned}
C^{((2,2))}_{(1,3)(1,3)(1,3)} &= -\tfrac{2\sqrt{2}}{\sqrt{3}} \ + \ \sqrt{6}\,\epsilon \ + \ O(\epsilon^2) \\
C^{((2,2))}_{(3,1)(3,1)(3,1)} &= \tfrac{2\sqrt{2}}{\sqrt{3}} \ + \ \sqrt{6}\,\epsilon \ + \ O(\epsilon^2) \\
C^{((2,2))}_{(3,3)(3,3)(3,3)} &= \tfrac{2}{\sqrt{3}} \qquad\quad + \ O(\epsilon^2) \\
C^{((2,2))}_{(3,3)(3,3)(1,3)} &= \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon \ + \ O(\epsilon^2) \\
C^{((2,2))}_{(3,3)(3,3)(3,1)} &= \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon \ + \ O(\epsilon^2) \\
C^{((2,2))}_{(3,3)(3,1)(1,3)} &= \tfrac{1}{\sqrt{3}} \qquad\quad + \ O(\epsilon^2)
\end{aligned}
\tag{A.19}
$$

We also need the ope of two generic primary boundary fields ϕ_i of weight h_i to a third:

$$\phi_1(x)\,\phi_2(y) = (x-y)^{\Delta_1}\,C\,\phi_3(y) \ + \ (x-y)^{\Delta_1+1}\,\frac{\Delta_2}{2h_3}C\,\phi'_3(y) \ + \ \ldots \ , \quad x > y$$

$$\text{where } \Delta_1 \equiv h_3 - h_2 - h_1 \ , \quad \Delta_2 \equiv h_3 + h_1 - h_2 \ , \quad C \equiv C_{12}{}^3 \ . \tag{A.20}$$

Substituting the structure constants for the $(2,2)$ boundary condition from equation (A.19) into equation (A.20) and its derivatives, and taking the limit $\epsilon \to 0$, we obtain the opes in the $c=1$ model (all for $x > y$). As an example, to obtain the ope of d_3 with itself, we consider first that of ϕ with itself, for $c < 1$:

$$
\begin{aligned}
\phi(x)\,\phi(y) \ = \ & (x-y)^{-4\epsilon^2} \ + \ \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon(x-y)^{1-2\epsilon}\,\psi(y) \ + \ \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon(x-y)^{1+2\epsilon}\,\bar{\psi}(y) \\
& + \ \tfrac{2}{\sqrt{3}}(x-y)^{-2\epsilon^2}\,\phi(y) \ + \ \tfrac{1}{\sqrt{3}}(x-y)^{1-2\epsilon^2}\,\phi'(y) \ + \ \ldots \ , \quad x > y \ ,
\end{aligned}
\tag{A.21}
$$

where we have dropped less singular terms and terms of order $O(\epsilon^3)$. Taking the limit $\epsilon \to 0$ of this equation, we recover the first eqn. of (A.23), and taking the limit of the x and y derivatives, we recover the last eqn. of (A.24), e.g.

$$
\begin{aligned}
d_3(x)\,d_3(y) = & \lim_{\epsilon \to 0} \tfrac{1}{4\epsilon^2}\,\phi'(x)\,\phi'(y) \\
= & \lim_{\epsilon \to 0} \tfrac{1}{4\epsilon^2}\Big(-4\epsilon^2(x-y)^{-2} \ + \ \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon(2\epsilon)(x-y)^{-1}\,\psi(y) \\
& + \ \tfrac{2\sqrt{2}}{\sqrt{3}}\epsilon(-2\epsilon)(x-y)^{-1}\,\bar{\psi}(y) \ + \ \tfrac{2}{\sqrt{3}}(-2\epsilon^2)(x-y)^{-2}\,\phi(y) \ + \ \ldots \Big) \\
= & -\tfrac{1}{(x-y)^2}(1 + \tfrac{1}{\sqrt{3}}\phi(y)) \ + \ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}(\psi(y) - \bar{\psi}(y)) \ + \ O(1) \ , \quad x > y \ .
\end{aligned}
\tag{A.22}
$$

The opes involving the field ϕ are both regular and exact:

$$\phi(x)\ \phi(y) = \quad 1 \quad + \quad \tfrac{2}{\sqrt{3}}\phi(y)\,,$$

$$\phi(x)\ \psi(y) = \tfrac{1}{\sqrt{3}}\bar\psi(y)\ -\ \sqrt{\tfrac{2}{3}}\,d_3(y)\,,$$

$$\phi(x)\ \bar\psi(y) = \tfrac{1}{\sqrt{3}}\psi(y)\ -\ \sqrt{\tfrac{2}{3}}\,d_3(y)\,,$$

$$\psi(x)\ \phi(y) = \tfrac{1}{\sqrt{3}}\bar\psi(x)\ +\ \sqrt{\tfrac{2}{3}}\,d_3(x)\,, \qquad\qquad \text{(A.23)}$$

$$\bar\psi(x)\ \phi(y) = \tfrac{1}{\sqrt{3}}\psi(x)\ +\ \sqrt{\tfrac{2}{3}}\,d_3(x)\,,$$

$$\phi(x)\ d_3(y) = -\sqrt{\tfrac{2}{3}}\,\psi(y) - \sqrt{\tfrac{2}{3}}\,\bar\psi(y) + \tfrac{1}{\sqrt{3}}\,d_3(y)\,,$$

$$d_3(x)\ \phi(y) = \sqrt{\tfrac{2}{3}}\,\psi(x)\ +\ \sqrt{\tfrac{2}{3}}\,\bar\psi(x) + \tfrac{1}{\sqrt{3}}\,d_3(x)\,.$$

Note that the structure constants of these fields are no longer cyclically symmetric – for example $C_{d_3\,\phi\,\psi} = -C_{\phi\,d_3\,\psi} = \sqrt{2/3}$. The opes of the weight one fields are more complicated; again we give them for $x > y$:

$$\psi(x)\ \psi(y) = \quad \tfrac{1}{(x-y)^2} \quad - \quad \tfrac{2\sqrt 2}{\sqrt 3}\tfrac{1}{(x-y)}\,\psi(y) + \ O(1)\,,$$

$$\bar\psi(x)\ \bar\psi(y) = \quad \tfrac{1}{(x-y)^2} \quad + \quad \tfrac{2\sqrt 2}{\sqrt 3}\tfrac{1}{(x-y)}\,\bar\psi(y) + \ O(1)\,,$$

$$\psi(x)\ \bar\psi(y) = \tfrac{1}{\sqrt 3}\tfrac{1}{(x-y)^2}\,\phi(y)\ -\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,d_3(y) + \ O(1)\,,$$

$$\bar\psi(x)\ \psi(y) = \tfrac{1}{\sqrt 3}\tfrac{1}{(x-y)^2}\,\phi(y)\ +\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,d_3(y) + \ O(1)\,,$$

$$\psi(x)\ d_3(y) = \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)^2}\,\phi(y)\ -\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,\bar\psi(y)\ -\ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}\,d_3(y)\ + O(1)\,, \qquad \text{(A.24)}$$

$$\bar\psi(x)\ d_3(y) = \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)^2}\,\phi(y)\ +\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,\psi(y)\ +\ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}\,d_3(y)\ + O(1)\,,$$

$$d_3(x)\ \psi(y) = -\sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)^2}\,\phi(y) +\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,\bar\psi(y)\ -\ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}\,d_3(y)\ + O(1)\,,$$

$$d_3(x)\ \bar\psi(y) = -\sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)^2}\,\phi(y) -\ \tfrac{2}{\sqrt 3}\tfrac{1}{(x-y)}\,\psi(y)\ +\ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}\,d_3(y)\ + O(1)\,,$$

$$d_3(x)\ d_3(y) = -\ \tfrac{1}{(x-y)^2}\big(1 + \tfrac{1}{\sqrt 3}\phi(y)\big)\ +\ \sqrt{\tfrac{2}{3}}\tfrac{1}{(x-y)}\,(\psi(y) - \bar\psi(y))\ +\ O(1)\,.$$

A.3   The $c \to 1$ limit of the $(2,p)$ boundary condition

There are two fields of weight zero on the $c \to 1$ limit of the $(2,p)$ b.c., namely ϕ_{11} and ϕ_{33}. From section A.1.2 we have

$$C^{((2p)(2p)(2p))(33)}_{(33)(33)} = \frac{2}{\sqrt{p^2-1}}\,, \qquad\qquad \text{(A.25)}$$

which gives the ope

$$\phi_{33}(x)\ \phi_{33}(y) = \phi_{11}(y)\ +\ \tfrac{2}{\sqrt{p^2-1}}\phi_{33}(y)\,. \qquad\qquad \text{(A.26)}$$

From this we deduce that the projectors are

$$P_a \;=\; \tfrac{p-1}{2p}\phi_{11} \;+\; \tfrac{\sqrt{p^2-1}}{2p}\phi_{33}\,, \tag{A.27}$$

$$P_b \;=\; \tfrac{p+1}{2p}\phi_{11} \;-\; \tfrac{\sqrt{p^2-1}}{2p}\phi_{33}\,. \tag{A.28}$$

These can be inverted to give

$$\phi_{11} \;=\; P_a \;+\; P_b\,, \tag{A.29}$$

$$\phi_{33} \;=\; \sqrt{\tfrac{p+1}{p-1}}\,P_a \;-\; \sqrt{\tfrac{p-1}{p+1}}\,P_b\,. \tag{A.30}$$

In the case of the (22) boundary, P_a projects onto the $\widehat{(1)}$ boundary and P_b on the $\widehat{(3)}$ boundary, and in general P_a projects onto the $\widehat{(p-1)}$ boundary and P_b on the $\widehat{(p+1)}$ boundary,

A.4   The $c \to 1$ limit of the $(3,p)$ boundary condition

There are three fields of weight zero on the $c \to 1$ limit of the $(3,p)$ b.c., namely ϕ_{11}, ϕ_{33} and ϕ_{55}. From section A.1.2 we have

$$A \equiv C^{((3p)(3p)(3p))(33)}_{(33)(33)} \;=\; \sqrt{\tfrac{3}{2}}\tfrac{1}{\sqrt{p^2-1}}\,, \qquad B \equiv C^{((3p)(3p)(3p))(55)}_{(33)(33)} \;=\; \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{p^2-4}{p^2-1}}\,,$$

$$C \equiv C^{((3p)(3p)(3p))(33)}_{(55)(55)} \;=\; 3\sqrt{\tfrac{3}{2}}\tfrac{1}{\sqrt{p^2-1}}\,, \qquad D \equiv C^{((3p)(3p)(3p))(55)}_{(55)(55)} \;=\; -\tfrac{1}{\sqrt{2}}\tfrac{p^2-16}{\sqrt{(p^2-4)(p^2-1)}}\,. \tag{A.31}$$

which gives the opes

$$\phi_{33}\,\phi_{33} \;=\; \phi_{11} \;+\; A\phi_{33} \;+\; B\phi_{55}\,, \tag{A.32}$$

$$\phi_{33}\,\phi_{55} \;=\; B\phi_{33} \;+\; C\phi_{55}\,, \tag{A.33}$$

$$\phi_{55}\,\phi_{55} \;=\; \phi_{11} \;+\; C\phi_{33} \;+\; D\phi_{55}\,. \tag{A.34}$$

From these we deduce that the projectors are

$$P_a \;=\; \tfrac{p-2}{3p}\phi_{11} \;+\; \tfrac{p-2}{p}\sqrt{\tfrac{p+1}{6(p-1)}}\phi_{33} \;+\; \tfrac{1}{3p}\sqrt{\tfrac{(p+1)(p^2-4)}{2(p-1)}}\phi_{55}\,, \tag{A.35}$$

$$P_b \;=\; \tfrac{1}{3}\phi_{11} \;+\; \sqrt{\tfrac{2}{3(p^2-1)}}\phi_{33} \;-\; \tfrac{1}{3}\sqrt{\tfrac{2(p^2-4)}{(p^2-1)}}\phi_{55}\,, \tag{A.36}$$

$$P_c \;=\; \tfrac{p+2}{3p}\phi_{11} \;-\; \tfrac{p+2}{p}\sqrt{\tfrac{p-1}{6(p+1)}}\phi_{33} \;+\; \tfrac{1}{3p}\sqrt{\tfrac{(p-1)(p^2-4)}{2(p+1)}}\phi_{55}\,. \tag{A.37}$$

These can be inverted to give

$$\phi_{11} \;=\; P_a \;+\; P_b \;+\; P_c\,, \tag{A.38}$$

$$\phi_{33} \;=\; \sqrt{\tfrac{3(p+1)}{2(p-1)}}\,P_a \;+\; \sqrt{\tfrac{6}{p^2-1}}\,P_b \;-\; \sqrt{\tfrac{3(p-1)}{2(p+1)}}\,P_c\,, \tag{A.39}$$

$$\phi_{55} \;=\; \sqrt{\tfrac{(p+2)(p+1)}{2(p-2)(p-1)}}\,P_a \;-\; \sqrt{\tfrac{2(p^2-4)}{p^2-1}}\,P_b \;+\; \sqrt{\tfrac{(p-2)(p-1)}{2(p+2)(p+1)}}\,P_c\,. \tag{A.40}$$

In the case of the (33) boundary, P_a, P_b and P_c project onto the $\widehat{(1)}$, $\widehat{(3)}$ and $\widehat{(5)}$ boundaries respectively.

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